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COMMENT

# On Smoluchowski's coagulation equation

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**Abstract.** We give a short proof of the global existence and uniqueness of solutions of Smoluchowski's coagulation equation with monodisperse initial conditions.

## 1. Introduction

Leyvraz and Tschudi (1981, hereafter referred to as LT) have considered the system of equations

$$\dot{c}_j = \tfrac{1}{2}\sum_{k=1}^{j-1} r_k r_{j-k} c_k c_{j-k} - r_j c_j \sum_{k=1}^{\infty} r_k c_k$$
$$c_j(0) = \delta_{j1} \qquad r_k = Ak + B \tag{1.1}$$

(LT, p 3394). They find solutions global in time, first for $B \neq 0$ and then by a limiting process for $B = 0$. This is in contrast with McLeod (1964, hereafter referred to as ML2), where it is stated (for the case $B = 0$ and $A = 1$) that there is no reasonable solution for a wider range of $t$ values than $0 \leqslant t \leqslant 1$. The reasons for this discrepancy are their different interpretations of the term 'solution of (1.1)'. McLeod demands $\Sigma_{k=1}^{\infty} k^2 c_k$ to be uniformly absolutely convergent on the solution interval, Leyvraz and Tschudi merely that (1.1) makes sense pointwise—that $\Sigma_{k=1}^{\infty} kc_k$ is convergent at each point of the interval.

In this comment I give a short proof of the global uniqueness (and existence) of solutions of (1.1) with $B = 0$ in the sense of (LT). The proof is based on a suggestion of T A Bak. It uses the methods of McLeod (1962, hereafter referred to as ML1), a more thorough investigation of the series $\Sigma_{k=1}^{\infty} k^{k-1} z^k / k!$ and the fact that continuity of $\Sigma_{k=0}^{\infty} kc_k$ follows from the equation and continuous induction.

## 2. The theorem

We consider the coagulation equation with monodisperse initial data:

$$\dot{c}_k = -kc_k \sum_{j=1}^{\infty} jc_j + \tfrac{1}{2}\sum_{j=1}^{k-1} j(k-j) c_j c_{k-j} \qquad c_k(0) = \delta_{1k}. \tag{2.1}$$

We want to solve (2.1) for all $t$. Let

$$\phi^0(t) = \begin{cases} t & \text{when } 0 \leqslant t \leqslant 1 \\ 1 + \log t & \text{when } 1 < t < \infty \end{cases} \tag{2.2}$$
$$x_k^0(t) = k^{k-2} t^{k-1} / k! \qquad \text{when } 0 \leqslant t < \infty \qquad k = 1, 2, \ldots$$

thus $x_k^0(t)$ is the unique solution for all times of the system

$$\dot{x}_k^0(t) = \tfrac{1}{2}\sum_{j=1}^{k-1} j(k-j)x_j^0(t)x_{k-j}^0(t) \qquad x_k^0 = \delta_{1k}. \tag{2.3}$$

Before considering (2.1) we prove a technical lemma, which essentially is a precise statement of some facts noted in (MLI, pp 122–3).

*Lemma 2.1.* The power series

$$\psi(z) = \sum_{k=1}^{\infty} k^{k-1}z^k/k!$$

is uniformly convergent in the disc $\{z \mid |z| \leq e^{-1}\}$ and divergent outside. The sum is continuous in the disc, $C^\infty$ in the interior and satisfies

$$\psi(z) = z\, e^{\psi(z)} \qquad \text{and} \qquad |\psi(z)| \leq 1 \qquad \text{when } |z| \leq e^{-1}$$

$$\psi(w\, e^{-w}) = w \qquad \text{when } |w| \leq 1.$$

Moreover, $\psi$ is increasing on $[0, e^{-1}]$ with $\psi(0) = 0$ and $\psi(e^{-1}) = 1$.

*Proof.* By Stirling's formula, $k^{k-1}z^k/k! \sim k^{-3/2}(z\,e)^k$ as $k \to \infty$, which implies the assertions on the convergence, and hence the assertions on continuity and differentiability of $\psi$. To prove (2.1), note that when $|w| = 1$ and $|z| < e^{-1}$ then $|z\, e^w| = |z|\, e^{\Re w} < 1$; thus by Rouché's theorem, the equation

$$w - z\, e^w = 0 \qquad (\text{i.e. } z = w\, e^{-w})$$

has exactly one solution $w$ with $|w| < 1$ when $|z| < e^{-1}$, since this is true when $z = 0$. By the residue theorem, that solution is given by

$$\frac{1}{2\pi i}\oint_{|w|=1} w\frac{(d/dw)(w - z\, e^w)}{w - z\, e^w}\, dw = \frac{1}{2\pi i}\oint_{|w|=1} (1 - z\, e^w)\sum_{k=0}^{\infty}\left(\frac{z\, e^w}{w}\right)^k dw$$

$$= \sum_{k=1}^{\infty} k^{k-1}z^k/k! = \psi(z).$$

Thus $\psi(z) = z\, e^{\psi(z)}$ and $|\psi(z)| \leq 1$ when $|z| < e^{-1}$, hence by continuity for $|z| \leq e^{-1}$. Since $|w\, e^{-w}| < e^{-1}$ when $|w| < 1$, $\psi(w\, e^{-w}) = w$ then, hence for $|w| \leq 1$ by continuity. The last assertion follows immediately.

*Theorem 2.2.* (i) If we define $c_k$ on $[0, \infty[$ by

$$c_k(t) = x_k^0(t)\, e^{-k\phi^0(t)} \qquad k = 1, 2, \ldots \tag{2.4}$$

then all $c_k$ are continuously differentiable for $0 \leq t < \infty$

$$\sum_{k=1}^{\infty} kc_k(t) = \begin{cases} 1 & \text{when } 0 \leq t \leq 1 \\ 1/t & \text{when } 1 < t < \infty \end{cases}$$

and (2.1) is satisfied for all $t$.

(ii) If $\theta > 0$ and $c_k$, $k = 1, 2, \ldots$, are continuously differentiable functions on $[0, \theta[$ such that $\Sigma_{k=1}^{\infty} kc_k(t)$ is convergent for all $t \in [0, \theta[$ and $(c_k)$ satisfies (2.1) on $[0, \theta[$; then $(c_k)$ is given by (2.4) for $t \in [0, \theta[$.

*Proof.* Assume $\theta$ and $(c_k)$ given as in (ii). Note that $c_1(0)=1>0$ and let $\theta_1$ be the largest positive number $\leq\theta$ such that $c_1(t)>0$ when $0\leq t<\theta_1$ (we shall prove that $\theta_1=\theta$). By (2.1) for $\dot{c}_1$, $\Sigma_{j=1}^{\infty} jc_j(t)$ is continuous on $[0, \theta_1[$, thus we may define

$$\phi(t)=\int_0^t \sum_{j=1}^{\infty} jc_j(u)\,\mathrm{d}u \qquad \text{and} \qquad x_k(t)=c_k(t)\,\mathrm{e}^{k\phi(t)} \qquad \text{on } [0, \theta_1[.$$

Then $\phi$ is continuously differentiable, $\phi(0)=0$, $\dot{\phi}(0)=1$ and by (2.1), $x_k(0)=\delta_{k1}$ and

$$\dot{x}_k(t)=\left(\dot{c}_k(t)+kc_k(t)\sum_{j=1}^{\infty} jc_j(t)\right)\mathrm{e}^{k\phi(t)}=\tfrac{1}{2}\sum_{j=1}^{k-1} j(k-j)x_j(t)x_{k-j}(t).$$

It follows that $x_k(t)=x_k^0(t)$ on $[0, \theta_1[$, and thus

$$t\dot{\phi}(t)=t\sum_{k=1}^{\infty} kc_k(t)=t\sum_{k=1}^{\infty} kx_k^0(t)\,\mathrm{e}^{-k\phi(t)}=\sum_{k=1}^{\infty} k^{k-1}(t\,\mathrm{e}^{-\phi(t)})^k/k!. \tag{2.5}$$

By assumption this series is convergent, so the lemma implies

$$0\leq t\,\mathrm{e}^{-\phi(t)}\leq e^{-1} \qquad \text{and} \qquad t\dot{\phi}(t)=\psi(t\,\mathrm{e}^{-\phi(t)}) \qquad \text{on } [0, \theta_1[. \tag{2.6}$$

We must prove that $\phi(t)=\phi^0(t)$ on $[0, \theta_1[$. To do that, let $\theta_2$ be the largest positive number $\leq\theta_1$ such that $t\,\mathrm{e}^{-\phi(t)}<\mathrm{e}^{-1}$ when $0\leq t<\theta_2$. By (2.6) and the lemma, $\phi$ is $C^{\infty}$ on $]0, \theta_2[$, $t\dot{\phi}(t)<1$ there and we have

$$\dot{\phi}(t)=\frac{1}{t}\psi(t\,\mathrm{e}^{-\phi(t)})=\frac{1}{t}\,t\,e^{-\phi(t)}\exp[\psi(t\,\mathrm{e}^{-\phi(t)})]=\mathrm{e}^{-\phi(t)+t\dot{\phi}(t)}$$

when $0<t<\theta_2$. By differentiation, $\ddot{\phi}(t)=\ddot{\phi}(t)t\dot{\phi}(t)$. Hence $\ddot{\phi}(t)=0$ when $0<t<\theta_2$ and therefore $\phi(t)=t$ on $[0, \theta_2[$. Since $t\,\mathrm{e}^{-t}<\mathrm{e}^{-1}$ when $t<1$, we get $\theta_2=\min\{1, \theta_1\}$.

Now assume $\theta_1>1$; then by (2.6) and the lemma, $\phi(t)\geq 1+\log t$ and $\dot{\phi}(t)=(1/t)\psi(t\,\mathrm{e}^{-\phi(t)})\leq(1/t)$ on $]0, \theta_1[$, hence when $1\leq t<\theta_1$:

$$1+\log t\leq\phi(t)\leq\phi(1)+\int_1^t \frac{1}{u}\,\mathrm{d}u\leq 1+\log t$$

and thus $\phi(t)=\phi^0(t)$ when $0\leq t<\theta_1$. To prove (ii) it remains to be proved that $\theta_1=\theta$, but $c_1(t)=x_1^0(t)\,\mathrm{e}^{-\phi^0(t)}$ when $0\leq t\leq\theta_1$, and $x_1^0(\theta_1)\,\mathrm{e}^{-\phi^0(\theta_1)}>0$, so $\theta_1=\theta$ by the definition of $\theta_1$.

To prove (i), note that with $c_k$ defined by (2.4), the computation (2.5) and the lemma shows that

$$\sum_{j=1}^{\infty} jc_j(t)=(1/t)\psi(t\,\mathrm{e}^{-\phi^0(t)})=\begin{cases}1 & \text{when } 0\leq t\leq 1\\ 1/t & \text{when } 1<t<\infty\end{cases}$$
$$=\dot{\phi}^0(t).$$

Thus

$$\dot{c}_k(t)=-kc_k(t)\sum_{j=1}^{\infty} jc_j(t)+\dot{x}_k^0(t)\,\mathrm{e}^{-k\phi^0(t)}.$$

Using (2.3), (2.1) follows and the theorem is proved.

Note that without any significant change in the proof, the condition in (ii) that the $c_k$ be $C^1$ may be weakened to the $c_k$ being absolutely continuous (and the series convergent almost everywhere).

## References

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